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STABILIZATION OF THE STEADY-STATE MOTIONS OF MECHANICAL SYSTEMS WITH CYCLICAL COORDINATES*

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The stabilization of the steady-state motions of holonomic systems with cyclical coordinates is considered, in cases when it is not essentially required that the system be exponentially stable with respect to all the phase variables. It is shown that the stabilization can be simplified by applying controls (of the feedback type) to only some of the cyclical variables. The control signals applied to the other cyclical variables are then used only to preserve the initial value of the momentum. From the initial equations, a linear subsystem which includes the controlled cyclical variables is isolated, and the methods of general control theory are used to construct control signals for it such that it is asymptotically stable with respect to the phase variables. Stability with respect to all the phase variables of the initial system is established by reducing the problem to a special case. When the subsystem has low dimensionality, the control coefficients can be found analytically, and when the dimensionality is high, they can be found by a computer with standard mathematical software, using the method of Repin and Tret'yakov /1/. The stabilization of systems with cyclical coordinates by applying forces with respect to these coordinates was first considered in /2/, from the standpoint of Lyapunov's second method /3/, and from the standpoint of general control theory /1/. The control signals were taken to be cyclical pulses, and asymptotic stability with respect to the positional coordinates and the velocities was obtained; it was remarked that control by forces applied with respect to the cyclical coordinates is possible. In /4, 5/, the stabilization of the steady-state motions of holonomic systems by forces applied with respect to the cyclical coordinates was analyzed qualitatively.

1. Consider a mechanical system which is constrained by geometrical non-stationary

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couplings under the action of potential forces. The position of the system is defined by the generalized coordinates $q_1, \ldots, q_n$, of which $n-k$ are cyclical, which the kinetic energy is

$$T = T_2 + T_1 + T_0$$
$$T_2 = {}^1/_2 \sum_{r,j=1}^{n} a_{rj}(q) q_j^{\cdot} q_r^{\cdot}, \quad T_1 = \sum_{j=1}^{n} b_j(q) q_j^{\cdot}$$
$$T_0 = T_0(q), \quad q' = (q_1, \ldots, q_k)$$

We assume that the subscripts run over the values $j = 1, \ldots, n;\ i = 1, \ldots, k;\ \alpha, \beta = k+1, \ldots, n;\ \mu = k+1, \ldots, k+m;\ \nu = k+m+1, \ldots, n.$

The state of the system is characterized by the Routh variables

$$q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_k^{\cdot}, p_{k+1}, \ldots, p_n$$
$$\left(p_\alpha = \frac{\partial T}{\partial q_\alpha^{\cdot}} = \sum_\beta a_{\alpha\beta} q_\beta^{\cdot} + \sum_i a_{\alpha i} q_i^{\cdot} + b_\alpha\right)$$

We introduce Routh's function

$$R(q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_k^{\cdot}, p_{k+1}, \ldots p_n) = T - \Pi(q) - \sum_\alpha p_\alpha q_\alpha^{\cdot}$$

($\Pi(q)$ is the system potential energy).

We write the equations of motion in Routh's form

$$\frac{d}{dt}\frac{\partial R}{\partial q_i^{\cdot}} - \frac{\partial R}{\partial q_i} = 0, \quad \frac{dp_\alpha}{dt} = 0$$

This system admits of cyclical integrals, and under certain initial conditions, can perform the stationary motions

$$q_i = q_i^{\circ} = \text{const}, \quad q_i^{\cdot} = 0, \quad p_\alpha = \delta_\alpha = \text{const} \tag{1.1}$$

Let us consider the possible stabilization to a first approximation of the stationary motion (1.1) by linear control signals applied with respect to m cyclical coordinates, where $(m \leqslant n-k)$, and let us find the type of stability obtained.

We form the equations of the disturbed motion in the light of the disturbances of the cyclical momenta, by putting

$$q_i = q_i^{\circ} + x_i, \quad p_\mu = \delta_\mu + x_\mu, \quad p_\nu = \delta_\nu + f_\nu$$

To a first approximation we obtain /6/

$$Ax_1^{\cdot} + Gx_1 + \Gamma_1 r^{\cdot} + Cx + H_1 r + \Gamma_2 f^{\cdot} + H_2 f = \Phi(x, x_1, r, f) \tag{1.2}$$
$$x^{\cdot} = x_1, \quad r^{\cdot} = 0, \quad f^{\cdot} = 0$$

where $A, G, \Gamma_1, C, H_1, \Gamma_2, H_2$ are constant matrices, expressible in the usual way in terms of the matrix of coefficients of the kinetic energy and the generalized forces; $\Phi(x, x_1, r, f)$ are non-linear terms.

We apply linear control forces $u_\mu(x, x_1, r)$ with respect to the variables r_μ and consider the linear controlled subsystem of equations

$$Ax_1^{\cdot} + Gx_1 + \Gamma_1 r^{\cdot} + Cx + H_1 r = 0, \quad x^{\cdot} = x_1, \quad r^{\cdot} = u \tag{1.3}$$

The sufficient condition for the existence of a control $u = \| u_\mu(x, x_1, r)\|$, solving the stabilization problem for the zero solution of system (1.3) with minimization of the functional

$$I = \int_{t_0}^{\infty} [\Omega_1(x, x_1, r) + \Omega_2(u)]\, dt \tag{1.4}$$

where Ω_1 and Ω_2 are positive definite quadratic forms, is the condition

$$\text{rank } W = \text{rank }\{Q, PQ, \ldots, P^{2k+m-1}Q\} = 2k+m \tag{1.5}$$
$$Q = \left\| \begin{matrix} -A^{-1}\Gamma_1 \\ 0 \\ E_m \end{matrix} \right\|, \quad P = \left\| \begin{matrix} -A^{-1}G & -A^{-1}C & -A^{-1}H_1 \\ 0 & E_k & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

The optical control has the structure /1/

$$u^{\circ} = S_1 x_1 + S_2 x + S_3 r, \; S_1 = \| S_{\mu i}^1 \|, \; S_2 = \| S_{\mu i}^2 \|, \; S_3 = \| S \quad \| \tag{1.6}$$

Then, all the roots of the characteristic equation of the linear system (1.3) will have negative real parts.

The system of Eqs.(1.2) with controls u_μ° applied with respect to the variables r_μ, and its characteristic equation, take the form

$$Ax_1^\cdot + Gx_1 + \Gamma_1 r^\cdot + Cx + H_1 r + \Gamma_2 f^\cdot + H_2 f = \Phi(x, x_1, r, f) \tag{1.7}$$
$$x^\cdot = x_1, \quad r^\cdot = S_1 x_1 + S_2 x + S_3 r, \quad f^\cdot = 0$$
$$\lambda^{n-m-k} \det \left\| \begin{matrix} A\lambda^2 + G\lambda + C & \Gamma_1\lambda + H_1 \\ -S_1\lambda - S_2 & E_m\lambda - S_3 \end{matrix} \right\| = 0$$

where $n - m - k$ roots of the characteristic equation are zero, and the real parts of all the remaining roots are negative, by virtue of the choice of the control signals.

Consider the functions w and v, given implicitly by the system of equations

$$Cw(f) + H_1 v(f) + H_2 f = \Phi_1(f), \quad S_2 w(f) + S_3 v(f) = 0 \tag{1.8}$$

Here, $\Phi_1(f) = \Phi(w, 0, v, f)$ are non-linear terms which contain the free critical variables f.

Since the control (1.6) is chosen, it follows by the implicit function theorem that Eqs. (1.8) define in the neighbourhood of the point $w = 0, v = 0, f = 0$, the continuous functions w and v of the variable f. Making the Lyapunov substitution /7/ $x = \xi + w(f), r = \eta + v(f)$, we obtain from (1.7) a system of equations that satisfies all the conditions of the Lyapunov-Malkin theorem /7, 8/ on the stability in the special case of $n - m - k$ zero roots. We have thus proved the following theorem.

Theorem. If the matrix W given by (1.5) has rank $2k + m$, then the controls (1.6) which solve the optimum stabilization problem for the zero solution of the linear controlled subsystem (1.3) in the case when functional (1.4) is minimized, solve the stabilization problem for the zero solution of system (1.2).

In view of the proof of this theorem, the motion is asymptotically stable with respect to the variables ξ, η, x_1, though when we pass to the original variables, we obtain asymptotic stability with respect to x_1 and stability with respect to x, r, f.

Notes. 1°. Unlike the present (ideal) mechanical system, the dissipative forces which are usually actually present prevent the conservation of the uncontrolled cyclical momenta. We then need to introduce into the system supplementary drives which compensate the energy dissipation, e.g., of the feedback control type. At the same time, these drives can often be contructed quite simply, e.g., in the steady state the number of rotations of a gyroscope rotor can be held constant by applying a constant voltage to the drive motor and no feedback circuits are needed /9/, p.380.

In addition, there is a class of systems, see Example 3, for which there is virtually no dissipation, i.e., the conditions imposed on the problem are satisfied by virtue of the physico-mechanical and technical properties of the system.

2°. Note the case when $\Gamma_1 = 0$, e.g., when the initial system is gyroscopically disconnected with respect to the control variables, and the stationary motion considered is such that the matrix H_1 in it vanishes. It can be shown that this motion of the system, like that in /4/, cannot be stabilized by linear control signals applied with respect to the chosen part of the cyclical coordinates.

3°. In the case when the controls are applied with respect to all the cyclical variables $(m = n - k)$, the controlled subsystem (1.3) transforms into the system of equations of the first approximation of the initial system (1.2), while the controls obtained provide stabilization up to asymptotic stability in all the variables. Then, in the case of motion for which $H \neq 0$, there is still the possibility of stabilizing the stationary motion for gyroscopically disconnected systems, regardless of the presence of dissipative forces acting on the positional velocities.

2. In the general case, in problems of high dimensionality, we have remarked that the coefficients of the stabilizing signals can be evaluated on a computer. At the same time, in the case of objects where the controlled subsystem has low dimensionality, the problem can be entirely solved analytically. Let us solve, in general form, the problem of stabilizing the stationary motions of holonomic systems with a single positional, and several cyclical, coordinates.

Consider a mechanical system which is constrained by stationary couplings and is under the action of potential forces, and whose position is described by three generalized coordinates q_1, q_2, q_3, two of which, e.g., q_2 and q_3, are cyclical. Under certain initial conditions, the system can perform the stationary motions

$$q_1 = q_1^\circ = \text{const}, \quad p_2 = \delta_2 = \text{const}, \quad p_3 = \delta_3 = \text{const} \tag{2.1}$$

For the present mechanical system, the matrix G vanishes. As the controlled variable we take q_2. Eqs.(1.7) of the disturbed motion are

$$x_1^{\cdot} = cx + h_1 r + h_2 f + gu + \Phi(x, x_1, r, f) \qquad (2.2)$$
$$x^{\cdot} = x_1, \quad r^{\cdot} = u, \quad f^{\cdot} = 0$$

where c, h_1, h_2, g are constants.

The controlled linear subsystem is

$$x_1^{\cdot} = cx + h_1 r + gu, \quad x^{\cdot} = x_1, \quad r^{\cdot} = u \qquad (2.3)$$

In our problem, $\det W = h_1^2 - cg^2 = \Delta$.

Then, in the case $\Delta \neq 0$, by our theorem, the zero solution of system (2.2) is stabilized by the control

$$v^\circ = \nu_1 x_1 + \nu_2 x + \nu_3 r$$

up to asymptotic stability with respect to x_1, and in general, up to stability with respect to x, r, f.

We now note that, for a holonomic mechanical system which is described by two cyclical, and one positional, coordinates, and is under the action of a potential force in the neighbourhood of the stationary motion (2.1), Eq.(2.3) of the controlled linear subsystem is the same as the system of Eqs.(113.15) considered in the example of /1/ (p.506).

Then, taking the quality **criterion** of the transient in the form

$$I = \frac{1}{2}\int_0^\infty (x_1^2 + x^2 + r^2 + u^2)\, dt$$

and using (expressions (113.17) of /1/ for the coefficients of the optimal control, we have

$$\nu_1 = [g(a_3 + ca_1) - h_1(a_2 + c)]/\Delta, \quad \nu_2 = [g(a_2 + c) - h_1(a_3 + ca_1)]/\Delta \qquad (2.4)$$
$$\nu_3 = [gh_1(a_2 + c) - h_1^2 a_1 - a_3 g^2]/\Delta, \quad a_1 = 1 + \sqrt{2c + g^2 + 2a_3}$$
$$a_2 = a_1 - 1 + c^2 + h_1^2, \quad a_3 = \sqrt{c^2 + h_1^2}$$

Example 1. Let us examine the stabilization by our method of the motion of a heavy gyroscope with ideal universal suspension and vertical axis of rotation of the outer ring /10/.

Retaining the notation of /10/, we write the kinetic and potential energies and Routh's function for the gyroscope

$$T = {}^1/_2 [A(\theta^{\cdot 2} + \psi^{\cdot 2}\sin^2\theta) + C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta)^2] + {}^1/_2 A_2 \psi^{\cdot 2} + {}^1/_2 [A_1\theta^{\cdot 2} + B_1\psi^{\cdot 2}\sin^2\theta + C\psi^{\cdot 2}\cos^2\theta], \quad \Pi = -Pz_0\cos\theta$$
$$R = {}^1/_2 (A + A_1)\theta^{\cdot 2} - W(\theta)$$
$$W(\theta) = Pz_0\cos\theta + {}^1/_2 (p_\psi - p_\varphi\cos\theta)^2/Q(\theta) + {}^1/_2 p_\varphi^2/C$$
$$Q(\theta) = (A + B_1)\sin^2\theta + C_1\cos^2\theta + A_2$$

Corresponding to the cyclical coordinates φ and ψ we have the integrals

$$p_\varphi = C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta) = \text{const}$$
$$p_\psi = [Q(\theta) + C\cos^2\theta]\psi^{\cdot} + C\cos\theta\varphi^{\cdot} = \text{const}$$

The gyroscope can perform the stationary motions

$$\theta = \theta_0, \quad \theta^{\cdot} = 0, \quad p_\varphi = \delta_2 = \text{const}, \quad p_\psi = \delta_3 = \text{const}$$

given by the equation

$$\partial W/\partial\theta = \sin\theta\{[(p_\psi - p_\varphi\cos\theta)/Q(\theta)][p_\varphi - (p_\psi - p_\varphi\cos\theta)(A + B_1 - C_1)\cos\theta/Q(\theta)] - Pz_0\} = 0,$$

This equation is **satisfied** by three branches of stationary motions, two of which are the straight lines

$$\theta_0 = 0, \quad \theta_0 = \pi \qquad (2.5)$$

We shall apply control with respect to the **coordinate** φ. The coefficients c, h_1, h_2, g of Eqs. (2.2) are **given** by the expressions

$$c = \frac{1}{A + A_1}\frac{\partial^2 W}{\partial\theta^2}, \quad h_1 = \frac{1}{A + A_1}\frac{\partial^2 W}{\partial\theta\partial p_\varphi}, \quad h_2 = \frac{1}{A + A_1}\frac{1}{\partial\theta\partial p_\psi}, \quad g = 0$$

The coefficients h_1 and h_2 clearly vanish in the stationary motions (2.5), so that these motions cannot be stabilized to a first approximation by a control force **applied with respect** to either cyclical coordinate. The third branch of the stationary motions can be stabilized by a force whose coefficients are obtained by substituting for c, h_1, and g, in (2.4). We then obtain asymptotic stability with respect to $\theta^{\cdot}$ and stability with respect to θ, p_{φ}, and p_{ψ}.

It was shown in /2/ that the system is controllable in the problem of stabilizing the stationary motion when $\theta_0 \neq 0, \theta_0 \neq \pi$, and is not controllable if $\theta_0 = 0, \theta_0 = \pi$, when the two cyclical momenta are taken as the controls. Stabilization was there considered in the sense of achieving asymptotic stability with respect to θ and $\theta^{\cdot}$.

Example 2. When analyzing the stabilization to a first approximation of the trivial motions /4/ of a gyroscopically connected system /5/, there was a discussion of the stabilization of the trivial steady-state motions of a gyroscope with universal suspension, directed violation of the symmetry, and vertical axis of rotation of the outer ring. Apart from gravity forces, the gyroscope was acted on by a dissipative force with respect to the positional coordinate, and stabilizing moments were applied with respect to the two cyclical coordinates.

Retaining the notation of /5/, we consider the optimal stabilization to a first approximation of the steady-state motions of this gyroscope when there is no dissipation. We will show that stabilization is possible by a control applied only to one cyclical coordinate, and we write the coefficients of such a control. Introducing the cyclical moments $p_1 = \partial T/\partial \psi^{\cdot}$, $p_2 = \partial T/\partial \varphi^{\cdot}$, we write Routh's function for the system:

$$R = {}^1/_2 \left[a - \Delta^{-1}(\theta)\left(b_{22}c_1^2 - 2b_{12}c_1c_2 + b_{11}c_2^2\right)\right]\theta^{\cdot 2} + \Delta^{-1}(\theta)\left[(b_{22}c_1 - b_{12}c_2)\, p_1 + (b_{11}c_2 - b_{12}\, c_1)\, p_2\right]\theta^{\cdot} + (m_2 + m_3)\, g y_2 \sin\varepsilon\cos\theta - {}^1/_2\Delta^{-1}(\theta)\left[b_{22}p_1^2 - 2b_{12}p_1p_2 + b_{11}p_2^2\right]$$

$$\Delta(\theta) = b_{11}b_{22} - b_{12}^2, \quad b_{11} = b_{11}(\theta), \quad b_{12} = b_{12}(\theta), \quad c_1 = c_1(\theta)$$

The system admits, in particular, of the motion

$$\theta_0 = 0, \quad p_1 = \delta_1 = \mathrm{const}, \quad p_2 = \delta_2 = \mathrm{const} \tag{2.6}$$

In the neighbourhood of (2.6), the coefficients of Eqs.(1.7) of the disturbed motion are

$$A = a - \Delta^{-1}(0)\left[b_{22}c_1^2(0) - 2b_{12}(0)\, c_1(0)\, c_2 - b_{11}(0)\, c_2\right]$$
$$C = (m_3 + m_2)\, g y_2 \sin\varepsilon + {}^1/_2 \{\Delta^{-1}(0)\left[(V_1 + 2V_3)\,\delta_1^2 - 2V_2\delta_1\delta_2\right\} - \Delta^{-2}(0)\left[(V_1 + 2V_3)\, A_0 + 2V_2(V_4 - V_2)\right] d_0\}, \quad G = 0$$
$$H_1 = 0, \quad H_2 = 0, \quad \Gamma_1 = \Delta^{-1}(0)\left[b_{22}c_1(0) - b_{12}(0)\, c_2\right]$$
$$\Gamma_2 = \Delta^{-1}(0)\left[b_{12}(0)\, c_1(0) + b_{11}(0)\, c_2\right], \quad V_2 = A_0\sin\varepsilon$$
$$V_1 = \left[G_2 + (B_0 - A_0)\,\lambda\mu + m_3x_1y_1\right]\sin 2\varepsilon$$
$$V_3 = \left[B_2 - C_2 + (A_0 - B_0)\,\mu^2 + m_3(z_1^2 - y_1^2)\right]\sin^2\varepsilon$$
$$V_4 = A_0\cos\varepsilon, \quad d_0 = b_{22}\delta_1^2 - 2b_{12}(0)\,\delta_1\delta_2 + b_{11}(0)\,\delta_2^2$$

When the moment applied about the axis of the outer ring is used for stabilization, i.e., with respect to the ψ coordinate, the controllability condition is satisfied if $C\Gamma_1 \neq 0$. If the stabilizing moment is applied about the gyroscope rotor axis, i.e., with respect to coordinate φ, then the controllability condition is satisfied when $C\Gamma_2 \neq 0$. Obviously, these conditions are, in general, satisfied. The coefficients of the control $u_1 = \nu_{11}x_1 + \nu_{12}x + \nu_{13}y_1$, stabilizing the motion (2.6), are optimal in the sense of criterion I_1, while the coefficients of the control $u_2 = \nu_{21}x_1 + \nu_{22}x + \nu_{23}y_2$, stabilizing the same motions, which are optimal in the sense of criterion I_2, are found from (2.4) after replacing g in it by Γ_1/A and Γ_2/A respectively. Here,

$$I_\omega = \int_0^\infty (x_1^2 + x^2 + y_\omega^2 + u_\omega^2)\, dt, \quad \omega = 1,2$$

We then obtain asymptotic stability with respect to the positional velocity $\theta^{\cdot}$ and stability with respect to θ and the remaining velocities. Notice that this property can be found directly from the controls for stabilizing any steady-state motion (2.6), including the case of equilibrium; it is sufficient to put $\delta_1 = 0, \delta_2 = 0$ in the coefficients of the optimal control.

Note. In Examples 1 and 2 we have taken the case when there is no dissipation. If dissipative forces act on the gyroscope, there will always be a drive which compensates the action of resistance forces about the rotor axis and maintains (in the steady-state) a constant angular velocity of the rotation itself. It is therefore natural to apply the stabilizing moment with respect to the second cyclical coordinate ψ, i.e., about the axis of the outer frame. There are obviously no difficulties in principle when dissipation with respect to both $\psi^{\cdot}$ and $\theta^{\cdot}$ is taken into account.

Example 3. Consider the inertial motion of a space vehicle (SV), remote from the centre of attraction; the SV is a symmetric rigid body with a controlled flywheel, whose axis of rotation is directed along the axis of dynamic symmetry. After performing its last manoeuvre, the SV performs a residual motion about the centre of mass. We pose the problem of reducing this motion to a precession, by using the momentum produced by the flywheel.

The SV motion about its centre of mass will be described by the Euler angles θ, ψ, φ and the angle α of rotation of the rotor relative to the body. The Lagrange function of the system, as a free steady-state gyrostat, is

$$L = {}^1/_2 A\theta^{\cdot 2} + {}^1/_2 A \sin^2\theta\psi^{\cdot 2} + {}^1/_2 C(\varphi^\cdot + \psi^\cdot \cos\theta)^2 + \psi^\cdot\alpha^\cdot J\cos\theta + J\alpha^\cdot\varphi^\cdot + {}^1/_2 J\alpha^{\cdot 2}$$

Here, $A = B$, and C are the principal central moments of inertia of the **gyrostat, and** J is the axial moment of inertia of the rotor.

We introduce the momenta

$$\partial L/\partial\psi^\cdot = A\sin^2\theta\psi^\cdot + C(\varphi^\cdot + \psi^\cdot\cos\theta)\cos\theta + J\cos\theta\alpha^\cdot = p_1, \quad \partial L/\partial\varphi^\cdot = C(\varphi^\cdot + \psi^\cdot\cos\theta) + J\alpha^\cdot = p_2, \quad \partial L/\partial\alpha^\cdot = \psi^\cdot J\cos\theta + J\varphi^\cdot + J\alpha^\cdot = p_3$$

and Routh's function

$$R = \frac{1}{2}A\theta^{\cdot 2} - \frac{1}{2}\left[\frac{(p_1 - p_2\cos\theta)^2}{A\sin^2\theta} + \frac{p_2^2}{C-J} - \frac{2p_2p_3}{C-J} + \frac{Cp_3^2}{J(C-J)}\right]$$

The equations of motion of the SV about its centre of mass

$$A\theta^{\cdot\cdot} = A^{-1}\sin^{-3}\theta\,(p_1 - p_2\cos\theta)(p_2 - p_1\cos\theta)$$
$$p_1^\cdot = 0, \quad p_2^\cdot = 0, \quad p_3^\cdot = 0 \tag{2.7}$$

admit of the stationary motion

$$\psi_0^\cdot = 0, \quad p_1 = C\varphi_0^\cdot\cos\theta_0 + J\cos\theta_0\alpha_0^\cdot = k_1 = \text{const}, \quad \sin\theta_0 \neq 0$$
$$p_2 = C\varphi_0^\cdot + J\alpha_0^\cdot = k_2 = \text{const}, \quad p_3 = J(\varphi_0^\cdot + \alpha_0^\cdot) = k_3 = \text{const}$$

We regard the problem of reducing the SV residual motion to a precessional motion as a problem of stabilizing the motion (2.6). Introducing the disturbances

$$\theta = \theta_0 + x, \quad p_1 = k_1 + y_1, \quad p_3 = k_3 + y_3$$
$$\theta^\cdot = x_1, \quad p_2 = k_2 + y_2 \tag{2.8}$$

and separating the first approximation, we obtain from (2.7):

$$x^\cdot = x_1, \quad x_1^\cdot = ax + by_1 + cy_2 + X(x, x_1, y_1, y_2) \tag{2.9}$$
$$y_1^\cdot = 0, \quad y_2^\cdot = u, \quad y_3^\cdot = -u$$
$$a = \frac{k_2^2}{A^2}, \quad b = \frac{k_2}{A^2\sin\theta_0}, \quad c = -\frac{\cos\theta_0 k_2}{A^2\sin\theta_0}$$

where u is the moment acting from the rotor onto the body (e.g., the reactive moment of the stator of the flywheel drive motor), and X are non-linear terms. We isolate the controlled subsystem.

$$x^\cdot = x_1, \quad x_1^\cdot = ax + cy_2, \quad y_2^\cdot = u$$

The controllability **condition** is satisfied for it if $c \neq 0$, i.e., with $\cos\theta_0 \neq 0$ and $C\varphi_0^\cdot + J\alpha_0^\cdot \neq 0$ there exists the linear control

$$u = m_1x + m_2x_1 + m_3y_2 \tag{2.10}$$

such that the non-zero roots of the characteristic equation of the system of Eqs.(2.9) of the first approximation have negative real parts. These roots correspond to the variables x, x_1, y_2.

In the complete system (2.9) we have the critical case of two zero roots. To reduce this case to a singular case, we have to make the replacement

$$x = \xi + v_1(y_1), \quad y_2 = \eta + v_2(y_1)$$

where the functions v_1, v_2 are given by the system of equations

$$av_1 + by_1 + cv_2 + X(v_1, 0, y_1, v_2) = 0, \quad m_1v_1 + m_3v_2 = 0$$

which is always solvable for v_1 and v_2 if the controllability condition holds.

In short, there is a linear controlling moment (2.10) which stabilizes the motion (2.8) up to asymptotic stability with respect to $\theta^\cdot$ and stability with respect to θ, p_1, p_2 and p_3. On introducing the quality criterion

$$\int_0^\infty (x^2 + x_1^2 + y_2^2 + u^2)\,dt$$

we can write **explicitly, in the same way as above** (compare (2.4)) the **coefficients** m_1, m_2, m_3 of this control.

Note that the controlled subsystem of four or five equations in this problem is an uncontrolled scalar control, even if it depends on all the phase variables of the problem.

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ON THE CONSTRUCTION OF A BOUNDED CONTROL IN OSCILLATORY SYSTEMS*

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The motion of a linear controlled system from any initial state to a given final state is considered when there are geometric constraints on the control. One way of constructing the control when there are no constraints is to use a control signal formed by a linear combination of natural motions of the uncontrolled system /1, 2/. In the present paper this control method is used when there are geometrical constraints on the control functions. Sufficient conditions are obtained, under which this control law solves the problem in finite time. The same approach is applied to a multifrequency system of linear oscillators (pendulums) which are scalarly controlled. The control law is obtained and the process time is estimated. The control is also found for a two-mass system which contains an oscillatory unit.

1. Formulation of the problem. Consider a linear controlled dynamic sytstem with a bounded control

$$x^{\cdot} = A(t)x + B(t)u \tag{1.1}$$

$$|u(t)| \leqslant a, \quad a > 0 \tag{1.2}$$

Here, x is the n-dimensional vector of phase coordinates, u is the m-dimensional control vector, $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices respectively, piecewise continuously dependent on time t, and a is a positive constant.

We shall construct the control $u(t)$ which satisfies the constraint (1.2) for $t \in [t_0, T]$ and moves the system from the initial state

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